SOME CONSIDERATIONS ON THE HOMOGRAPHIC SOLUTIONS OF THE THREE BODY PROBLEM

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This paper deals with the problem of homographic solutions in the case of three bodies.

A solution of the problem of three bodies is homographic(Wintner,"The Analitical Foundations of Celestial Mechanics" page 284) if the configuration of the three bodies is similar for every time t.

The coordinate system of reference needs not to be inertial. Then, we introduce an "heliocentric" coordinate system, with the origin placed in one of the three masses, say m_3.

The equations of motion, with arbitrary exponent α for the mutual distances between the three masses are:

$$\ddot{r}_1 = -a_{11} r_1 - a_{12} r_2$$
$$\ddot{r}_2 = -a_{22} r_2 - a_{21} r_1$$

where:

$$a_{11} = K^2\left(\frac{m_1+m_3}{r_1^{\alpha+1}} + \frac{m_2}{r_3^{\alpha+1}}\right)$$
$$a_{12} = K^2 m_2\left(\frac{1}{r_2^{\alpha+1}} - \frac{1}{r_3^{\alpha+1}}\right)$$
$$a_{22} = K^2\left(\frac{m_2+m_3}{r_2^{\alpha+1}} + \frac{m_1}{r_3^{\alpha+1}}\right)$$
$$a_{21} = K^2 m_1\left(\frac{1}{r_1^{\alpha+1}} - \frac{1}{r_3^{\alpha+1}}\right)$$

According to the above definition, an homographic solution implies (Wintner, page 284) the existence of a scalar $\rho(t) > 0$ the dilatation, and an orthogonal matrix of rotation A $A(t)$ such that for every t:

$$r = \rho A \overset{\circ}{r} \qquad \text{with} \qquad \rho(0) = 1 \qquad \overset{\circ}{A} = E \tag{2}$$

the superscript corresponds to a fixed date t°.

Since (Wintner,page 285) $\overset{\circ}{r}$ is constant, taking the two derivates of the first members in (1), and replacing (2) on the second members,and multiplying afterwards by A^{-1} we get:

$$
\begin{aligned}
\left[\rho^{\alpha}\ddot{\rho}\,E + 2\rho^{\alpha}\dot{\rho}\,\Omega + \rho^{\alpha+1}(\dot{\Omega}+\Omega^2)\right]\mathring{r}_1 &= -\mathring{a}_{11}\,E\,\mathring{r}_1 - \mathring{a}_{12}\,E\,\mathring{r}_2 \\
\left[\rho^{\alpha}\ddot{\rho}\,E + 2\rho^{\alpha}\dot{\rho}\,\Omega + \rho^{\alpha+1}(\dot{\Omega}+\Omega^2)\right]\mathring{r}_2 &= -a_{22}\,E\,\mathring{r}_2 - a_{21}\,E\,\mathring{r}_1
\end{aligned} \tag{3}
$$

where: $\Omega = A^{-1}\dot{A}$

and: $\dot{\Omega}+\Omega^2 = A^{-1}\ddot{A}$

Ω and $\dot{\Omega}$ are antisymetric matrices.

$\mathring{r}_1, \mathring{r}_2, \mathring{a}_{ij}$ are the corresponding values for r_1, r_2, a_{ij} at the fixed date t^o

E is the unitary matrix.

Now let us reduce to diagonal form the three matrices Ω, Ω^2 and $\dot{\Omega}$

The simultaneous reduction to a diagonal form of two matrices, in ex. Ω and $\dot{\Omega}$ requires they be commutable.

In general this does not happen, except if we consider some special cases.

If we choose, for instance, a rotation around the Z axis, the corresponding orthogonal matrix A will be:

$$
A = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Performing now the operation quoted in (3) and putting for the components of the column vectors $\mathring{r}_1$ and $\mathring{r}_2$:

$$
\mathring{r}_1 = \{\, \mathring{q}_1\ \mathring{q}_2\ \mathring{q}_3 \,\} \qquad \mathring{r}_2 = \{\, \mathring{q}_4\ \mathring{q}_5\ \mathring{q}_6 \,\}
$$

we shall have now only one non -diagonal matrix

$$
\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

which has the proper values: i, -i, 0. Its equivalent matrix is therefore

$$
\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

We now equate the corresponding elements of the matrices in the two sides of our equations. By equating afterwards real and imaginary terms of the resulting equations

$$
\begin{aligned}
\rho^{\alpha}\ddot{\rho}\,\mathring{q}_1 - \rho^{\alpha+1}\dot{\phi}^2\mathring{q}_1 &= -\mathring{a}_{11}\mathring{q}_1 - \mathring{a}_{12}\mathring{q}_4 \\
\rho^{\alpha}\ddot{\rho}\,\mathring{q}_2 - \rho^{\alpha+1}\dot{\phi}^2\mathring{q}_2 &= -\mathring{a}_{11}\mathring{q}_2 - \mathring{a}_{12}\mathring{q}_5 \\
\rho^{\alpha}\ddot{\rho}\,\mathring{q}_3 &= -\mathring{a}_{11}\mathring{q}_3 - \mathring{a}_{12}\mathring{q}_6 \\
\rho^{2}\dot{\phi} &= C_1
\end{aligned}
\qquad (4)
$$

$$
\begin{aligned}
\rho^{\alpha}\ddot{\rho}\,\mathring{q}_4 - \rho^{\alpha+1}\dot{\phi}^2\mathring{q}_4 &= -\mathring{a}_{22}\mathring{q}_4 - \mathring{a}_{21}\mathring{q}_1 \\
\rho^{\alpha}\ddot{\rho}\,\mathring{q}_5 - \rho^{\alpha+1}\dot{\phi}^2\mathring{q}_5 &= -\mathring{a}_{22}\mathring{q}_5 - \mathring{a}_{21}\mathring{q}_2 \\
\rho^{\alpha}\ddot{\rho}\,\mathring{q}_6 &= -\mathring{a}_{22}\mathring{q}_6 - \mathring{a}_{21}\mathring{q}_3 \\
\rho^{2}\dot{\phi} &= C_2
\end{aligned}
\qquad (5)
$$

If we consider an initial isosceles triangle, with $r_1 = r_2$ and $m_1 = m_2$ the compatibility of both systems of differencial equations requires that

$$\mathring{q}_1 = \mathring{q}_4 \qquad \mathring{q}_2 = \mathring{q}_5 \qquad \mathring{q}_3 = -\mathring{q}_6$$

$$\alpha = 3$$

This particular value of the exponent α evidently corresponds to the special case considered in the election of the matrix of rotation A, namely, a rotation around the Z axis.

THE PROBLEM IN THE PLANE

In this case the corresponding orthogonal matrix A is:

$$A = \begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix}$$

and now we have for $\mathring{r}_1$ and $\mathring{r}_2$:

$$\mathring{r}_1 = \{\mathring{q}_1\,\mathring{q}_2\} \qquad \mathring{r}_2 = \{\mathring{q}_4\,\mathring{q}_5\}$$

Then, in the systems (4) and (5) we must exclude the third equation.

Suppose that for the initial configuration we choose an isosceles triangle with $r_1=r_2$; $m_1=m_2$; $q_1=q_4$; $q_2=-q_5$ It is clear that each group of differential equations will be incompatble, unless we put:

$$r_1 = r_2 = r_3$$

which corresponds to an initial equilateral triangle.

Now we may take the initial coordinates in a quite arbitrary way. Consequently for all values of the exponent α we shall have:

$$\rho^{\alpha}\ddot{\rho} - \rho^{\alpha+1}\dot{\phi}^2 = -n^2$$

$$\rho^2\dot{\phi} = c$$

It is remarkable that the masses are arbitrary.

By means of other simple considerations we can also show the existence of homographic collinear solutions.

In the discussions we follow the method exposed in Moulton's "Introduction to Celestial Mechanics" (1930).

I am indebted to Prof. G.M.Dedebant for his kind assistance on this problem.

The present results agree with those obtained by Prof. Dedebant in an unpublished memoir on the same subject.

Discusión:

Cesco: Con referencia a la parte del resumen de su trabajo que aparece en el programa en que afirma "Para el caso tridimensional se confirma la demostración de Wintner, completando su trabajo de Banachiewitz sobre la existencia de soluciones con ley de atracción inversamente proporcional al cubo de las distancias mutuas" deseo expresar que, lamentablemente, el teorema de Wintner según el cual las soluciones homográficas, con dicha ley de atracción, serían isósceles, es erróneo. Aparte de la demostración teórica que he expuesto en la "Reunión Internacional de Astrometría y Mecánica Celeste", he de citar como "contraejemplo" el siguiente (ej. mi trabajo de 1959, Serie Astronómica T.XXV N° 2, pag.9).

Sea $m_o = 0.7$, $m_1 = 0.1$, $m_2 = 0.2$, $a_1 = 2$, $a_2 = 1$, $a_o = 2.136157$

Si bien en otro caso

$$a_{11} = 0.059605 \qquad a_{12} = 0.190395$$

$$a_{21} = 0.0014475 \qquad a_{22} = 0.904803$$

Las coordenadas baricéntricas iniciales son:

$$\xi_0 = -0.198048 \qquad \xi_1 = 1.789235 \qquad \xi_2 = -0.201450$$

$$\eta_0 = -0.222517 \qquad \eta_1 = 0.002662 \qquad \eta_2 = 0.777479$$

La dilatación es

$$\rho = \sqrt{2\sqrt{|c|}\ t + 1} \qquad |c| = 0.059279$$

y la solución vectorial respecto de una terna unitaria con origen en m_o y paralelo al sistema baricéntrico es

$$R_1 = \rho P_1 = \rho(1.987283\ \check{i} + 0.225179\ (\check{j}\cos\Omega\tau + \check{k}\ \mathrm{sen}\,\Omega\tau)$$

$$R_2 = \rho P_2 = \rho(-0.003402\ \check{i} + 0.999996\ (\check{j}\cos\Omega\tau + \check{k}\ \mathrm{sen}\,\Omega\tau)$$

donde $\Omega = 0.919701$ y $\tau = \dfrac{1}{2\sqrt{|c|}} \ln\left(2\sqrt{|c|}\,t + 1\right)$

Se trata de una solución <u>homográfica escalena</u>. Sin embargo, verifica las condiciones de Banachiewitz:

1° : Los tres puntos (ξ_i, η_i) del plano inicial no son colineales ni tales que $\alpha_0 = \alpha_1 = \alpha_2$

2° :

$$m_0\xi_0 + m_1\xi_1 + m_2\xi_2 = 0$$

$$m_0\eta_0 + m_1\eta_1 + m_2\eta_2 = 0$$

$$m_0\xi_0\eta_0 + m_1\xi_1\eta_1 + m_2\xi_2\eta_2 = 0$$

$$\frac{m_0\xi_0\eta_0}{\alpha_0^4} + \frac{m_1\xi_1\eta_1}{\alpha_1^4} + \frac{m_2\xi_2\eta_2}{\alpha_2^4} = 0$$

siendo $\alpha_0 = a_0 \qquad \alpha_1 = a_2 \qquad \alpha_2 = a_1$

Altavista:1) El Dr.Cesco no puede afirmar que ha refutado la demostración de Wintner porque en su trabajo no incluye ninguna revisación de esa demostracion.

2) No ha comprendido el sentido de mi planteo. He tratado el caso de las soluciones homográficas en el problema de los tres cuerpos y he se[illegible] la definición clásica (Moulton, Wintner, Kurth), poniendo:

$$r = \rho . A . \overset{\circ}{r}$$

ρ la dilatación; A matriz de rotación ortogonal; $\overset{\circ}{r}$ vector que contiene las coordenadas iniciales y es constante (Wintner:"The analytical foundations of celestial mechanics" pág. 285) y he considerado una rotación particular alrededor del eje Z.

Que ese vector es constante puede comprobarse tambien en Moulton:" Introduction to Celestial Mechanics" y en Kurth: "On Lagrange's triangular solution of the problem of three bodies".

3) De acuerdo a Wintner (pág.286) si se pone $A = E$, E matriz unidad, se tiene una dilatación pura:

$$r = \rho \overset{\circ}{r}$$

y la solución es homotética. No hay rotación.

4) Luego el planteo que hace el Dr.Cesco corresponde, de acuerdo a lo que expresa el texto de Wintner, a una solución homotética; y además los vectores P y Q de su trabajo "Sobre las soluciones homográficas en el problema de los tres cuerpos" son constantes y sus derivadas nulas.

5) Mis resultados coinciden con la afirmación de Wintner y con los conseguidos por el Prof. G.M.Dedebant en un trabajo realizado en el año 1960.